GNSS Solutions:

The differences in differencing

"GNSS Solutions" is a regular column featuring questions and answers about technical aspects of GNSS. Readers are invited to send their questions to the columnist, **Dr. Mark Petovello**, Department of Geomatics Engineering, University of Calgary, who will find experts to answer them. His e-mail address can be found with his biography at the conclusion of the column.

How does measurement differencing affect my computed position?

The short answer is, "That depends."

Standalone, or undifferenced, measurements obviously produce the poorest positioning performance. However, how we proceed from this most basic data processing approach may affect the computed solution. In this article, we look at the details of why this occurs.

Measurements

We will initially concern ourselves only with pseudorange measurements, although the logic will apply equally to carrier phase and Doppler (phase rate) measurements. To this end, we will use the following simplified measurement model for the pseudorange

$$\vec{P} = \vec{\rho} + b \cdot \vec{1} + \vec{E} + \vec{n} \quad (1)$$

where $\vec{P}$ is the vector of pseudorange measurements, $\vec{\rho}$ is the vector of geometric ranges between the receiver and the satellites, b is the (scalar) receiver clock bias already scaled into units of length using the speed of light, $\vec{1}$ is a column-vector of ones, $\vec{E}$ is the vector of systematic errors (orbit error, satellite clock error, and atmosphere), and $\vec{n}$ is the vector containing the stochastic errors (noise and multipath).

Computing Position

For the purpose of our discussion, we will only consider least-squares as a means of estimating the unknown parameters, $\vec{x}$. For the measurements in equation (1), the unknowns are the user position and the receiver clock bias.

Furthermore, because the measurements are non-linear functions of the unknowns, these are usually linearized to compute corrections to our current estimate of our unknown parameters, $\delta\vec{x}$. This is written as

$$\delta\vec{x} = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \vec{w} \quad (2)$$

where H is the Jacobian matrix, R is the covariance matrix of the observation errors, and $\vec{w}$ is the vector of measurement misclosures consisting of the actual measurements minus the predicted measurements.

Without loss of generality, we assume that our initial estimate of our unknowns – $\vec{\rho}$ (containing the user position) and b – is perfect. In this case, the misclosure vector only contains the error terms from equation , that is, $\vec{w} = \vec{E} + \vec{n}$.

Furthermore, for a given set of measurements, H and R are constant and, thus, the magnitude of these misclosures/errors dictate the corrections to our unknowns. However, as per our initial assumption, since the corrections are applied to a perfect initial estimate, they can really be interpreted as the errors in the estimated parameters.

Between-Receiver Differences

Differential GNSS (DGNSS) positioning is typically implemented by differencing measurements to common satellites observed at two receivers, one of which is typically occupying a known coordinate.

The unknown parameters in this case are essentially the same as before, except that we are now estimating the *relative* clock bias between the two receivers. (However, because the clock bias is not usually of interest, this factor is of no practical importance.)

Assuming again that our initial estimate of the unknown parameters is perfect, the misclosure vector is now equal to the *difference* in the errors at the two receivers, that is, $\vec{w} = \Delta\vec{E} + \Delta\vec{n}$

Differencing the systematic errors typically results in a smaller magnitude. In contrast, differencing the stochastic errors increases the standard deviation by $\sqrt{2}$. For all but extremely long separations between the two receivers the overall misclosure/error magnitude decreases relative to the undifferenced case.

Because the matrix product to the left of $\vec{w}$ on the right side of equation (2) is the same as in the undifferenced case*, the resulting position error is also reduced. This is the primary motivation for DGNSS processing.

As the signals received at two nearby stations have largely the same propagation paths, their propagation errors are similar. Correspondingly, the common errors between two points are removed when the measurements are differenced but without introducing any additional unknowns.

Alternatively, this algorithm can be implemented by transmitting corrections from one receiver to the other. Once the errors are applied to equation the errors are reduced. In effect, therefore, the only thing that changes relative to the undifferenced case is the level of error, thus resulting in better positioning.

Figure 1 shows the three-dimensional position error obtained using undifferenced and between-receiver single-differenced data over a six-hour period. As expected, the errors from the single difference solution are smaller than in the undifferenced case.

Between-Satellite Differences

Unlike between-receiver differences, between-satellite differences only require one receiver. In this case, measurements to different satellites are differenced pairwise. Since any two observations from a given receiver made at the same time have the same clock error, this term is eliminated. Correspondingly, the vector of unknown parameters only contains the receiver position.

Of course, the reduction in the number of unknowns occurred at the expense of one observation (i.e., N undifferenced measurements reduces to N-1 between-satellite single difference measurements); so, the question remains as to what happens to the position estimate.

In fact, the result will be the same as in the undifferenced case, provided that the covariance matrix of the observations is updated as follows

$$R_{\nabla} = BRB^T \qquad (3)$$

where the ∇ subscript denotes the between-satellite difference and B is a coefficient matrix used to form the differences as

$$\nabla\vec{P} = B\vec{P} \qquad (4)$$

The equivalence between the undifferenced and between-satellite single difference can be shown mathematically using the known form of H and B in equations (2) and (3) respectively, but the derivation is quite involved and will therefore not be repeated here. Instead, a heuristic argument is made.

Basically, the same measurements are being used to estimate the same position states; nothing has changed in this regard. The fact that we formed a linear combination of the measure-

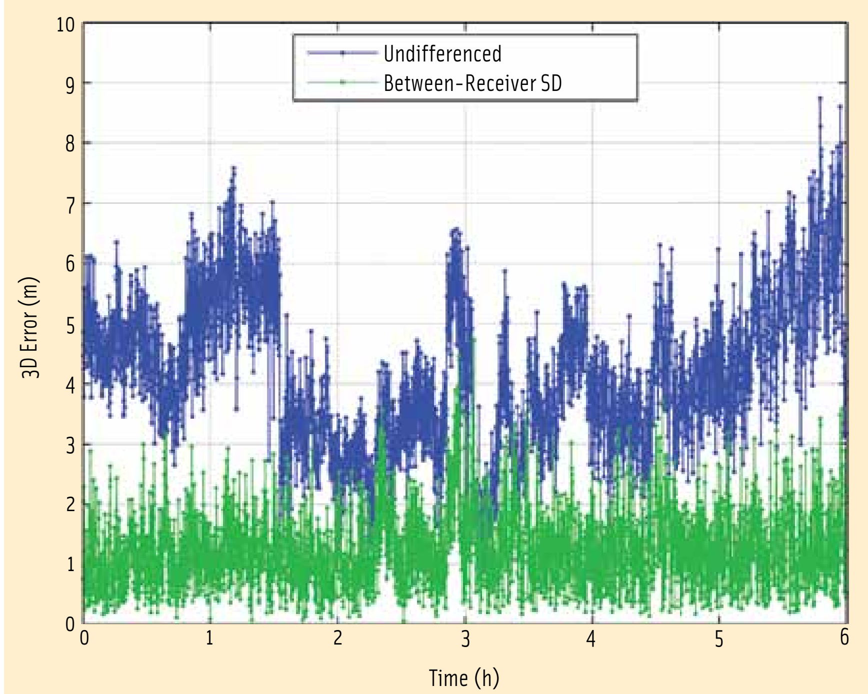

FIGURE 1 **Three-dimensional position errors over time using undifferenced and between-receiver single differenced data**

The common errors between two points [caused by signal propagation errors] are removed when the measurements are differenced but without introducing any additional unknowns.

* The covariance matrix of the observation error, R, is scaled relative to the undifferenced case to reflect the overall reduction in the magnitude of the errors. However, we can easily show that the product $(H^T R^{-1} H)^{-1} H^T R^{-1}$ is not sensitive to scaling R.

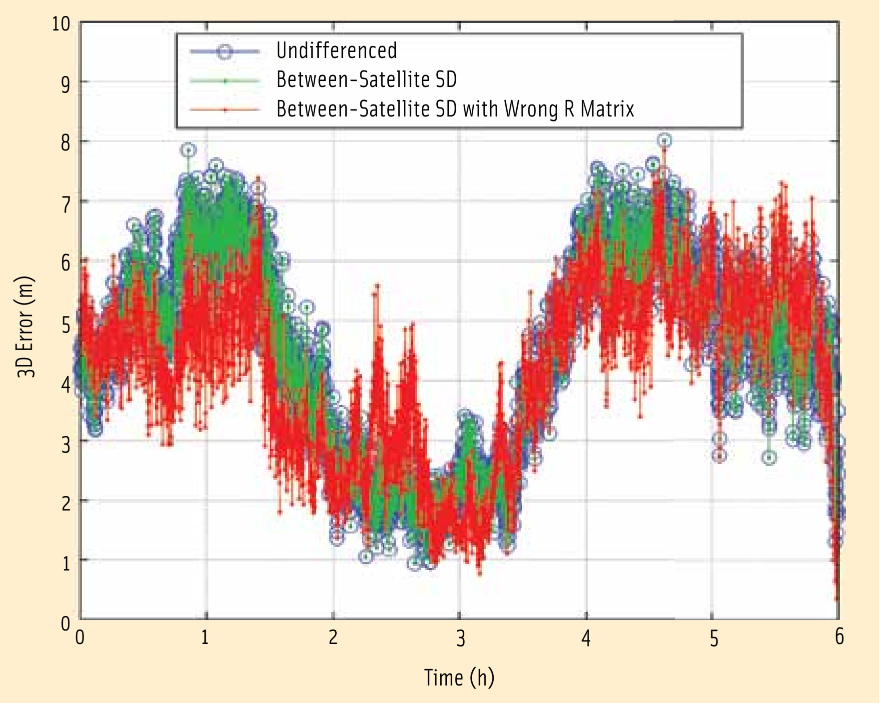

FIGURE 2 Three-dimensional position errors over time using undifferenced and between-satellite single differenced (SD) data. Also shown are the single difference results when the observation error covariance matrix is incorrectly specified.

ments prior to estimating the position — and that this combination happened to remove an unknown that we were not really interested in to start with — has no effect on the computed position.

Alternatively, we can think of removing the clock bias state as being offset by differencing the measurements using equation (4) and updating the covariance matrix of their errors via equation (3).

Figure 2 shows the results from processing the same data as in Figure 1. In this case, however, we omit the between-receiver single-difference solution and include two different between-satellite single-difference solutions.

The first single difference solution (green) is the "correct" solution in which the observation error covariance matrix was computed using equation (3), and as expected, the results are equivalent to the undifferenced case.

The second single difference solution (red) only uses the diagonal element from equation (3), and as can be seen, the results are now different. Furthermore, although not shown, in this case the estimated accuracy (covariance) of the solution will also be optimistic, which raises integrity issues.

Double Differencing

As is well known, double differences are the combination of the two aforementioned single differences. More specifically, double differences can be considered as between-satellite single differences of between-receiver single differences.

Analogous to how the between-satellite single difference did not affect the position solution relative to the undifferenced case, applying between-satellite single differencing to between-receiver single differences does not affect the solution either (relative to the between-receiver single difference solution). More simply, the position solution obtained with between-satellite single differences and double differences is the same.

A Look at Carrier Phase

Some might ask: given the last sentence, why is carrier-phase processing typically implemented using double-difference processing? Why not use single-difference processing?

The main reason is the desire to estimate the carrier phase ambiguities as integers. In order to do this with between-receiver single differences, we need to separate the receiver clock bias from the ambiguity states. Although this is possible — see, for example, the November/December 2006 GNSS Solutions contribution on Precise Point Positioning by Dr. Yang Gao, which has a related problem — it can take a significant amount of time.

In contrast, by forming the double difference, the clock bias is removed, thus allowing us to estimate the ambiguities independent of the clock bias. This is an overall simpler problem and allows for high accuracy positions to be computed more quickly.

Summary

Between-receiver differencing reduces the errors that are present in the measurements, thus producing a more accurate position estimate. Between-satellite differences do not change the position obtained (relative to the solution obtained using the observations being differenced), and are thus used only for practical reasons, such as simplifying the process of estimating the carrier phase ambiguities. IG

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