GNSS Solutions:

The math of ambiguity

"GNSS Solutions" is a regular column featuring questions and answers about technical aspects of GNSS. Readers are invited to send their questions to the columnist, **Dr. Mark Petovello**, Department of Geomatics Engineering, University of Calgary, who will find experts to answer them. His e-mail address can be found with his biography at the conclusion of the column.

What is the acquisition ambiguity function and how is it expressed mathematically?

One of the main tasks of a GNSS receiver is the acquisition of the signals-in-space (SISs) of all the satellites in view. This operation is based on the evaluation of a 2-D correlation function, called the *ambiguity function* (AF), which allows both the satellite detection and estimation of the received signal parameters, namely the code phase offset (code offset) and Doppler frequency/shift.

The AF is evaluated for each PRN code across all possible combinations of local code offset $\bar{\tau}$ and Doppler shift $\bar{f}_d$. This concept was well described in Michael Braasch's "GNSS Solutions" contribution in the March-April 2007 issue of *Inside GNSS*.

In order to decide on the presence or absence of the searched satellite, the maximum absolute value of the resulting AF is then compared with a predefined threshold. In fact, if the PRN code sought is present in the SIS the AF exhibits a well-defined peak, as shown in **Figure 1**.

The AF is generally evaluated in the digital section of the receiver using the following expression:

$$S_i(\bar{\tau}, \bar{f}_d) = \sum_{n=0}^{L-1} y_{IF}(nT_s)\, c_i(nT_s - \bar{\tau})\, e^{j2\pi(f_{IF}+\bar{f}_d)nT_s} \qquad (1)$$

where $y_{IF}(nT_s)$ is the sequence of samples of the analog signal $y_{IF}(t)$ at the front-end output, T_s is the sampling interval (the inverse of the sampling frequency of the analog-to-digital converter), $c_i(t)$ is the PRN code of the i-th satellite we are seeking, $\bar{\tau}$ is the local code offset, $\bar{f}_d$ is the local Doppler shift, L is the number of samples contained in the so called integration time $T_d(T_d = LT_s)$, and f_{IF} is the intermediate frequency (IF) of the carrier.

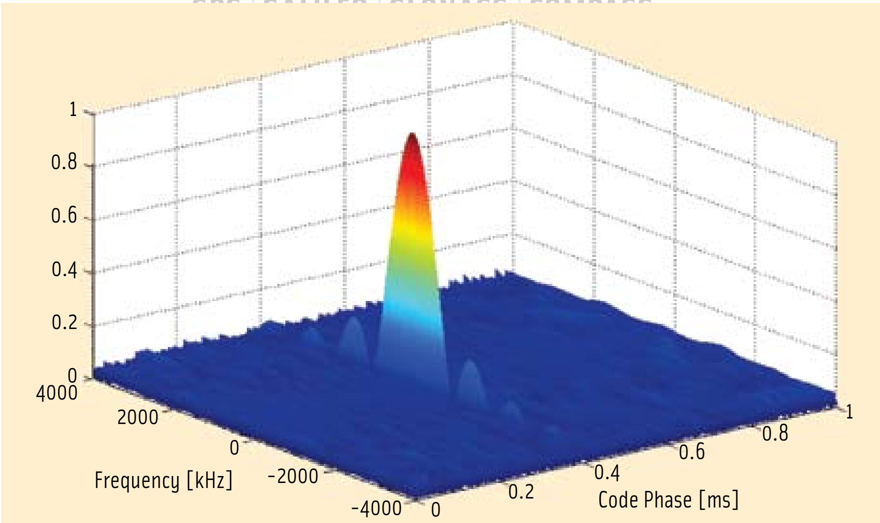


FIGURE 1 Sample of absolute value of AF in absence of noise for the L1 GPS C/A code signal

In the absence of noise the signal $y_{IF}(t)$ can be written as

$$y_{IF}(t)=\sum_{m=1}^{N}A_m d_m(t-\tau_m)c_m(t-\tau_m)\cos\left[2\pi\left(f_{IF}+f_{d,m}\right)t+\phi_m\right] \quad (2)$$

where
N is the number of satellites in view
A_m is the amplitude (> 0) of the m-th satellite signal
$d_m(t)$ is the data signal of the m-th satellite
$c_m(t)$ is the code of the m-th satellite
τ_m is the code offset of the m-th satellite
$f_{d,m}$ is the Doppler frequency of the m-th satellite signal
ϕ_m is the phase of the m-th satellite signal.

A closed-form expression that approximates the AF in the absence of noise is often given as

$$S_i\left(\bar{\tau},\bar{f_d}\right)\cong\frac{A_i}{2}e^{-j\phi_i}R_{c,i}(\Delta\tau)\mathrm{Sinc}(\Delta f T_d) \quad (3)$$

where the subscript i denotes the satellite sought, $\Delta\tau=\tau_i-\bar{\tau}$, $\Delta f=f_{d,i}-\bar{f_d}$, $R_{c,i}(\Delta\tau)$ is the normalized autocorrelation function of the PRN code $c_i(t)$, and Sinc(x) = $\sin(\pi x)/\pi x$.

At this point a number of questions arise. What is the validity region of this approximation in the plane $(\Delta\tau,\Delta f)$? Is it possible to have a closed-form expression valid in the whole plane $(\Delta\tau,\Delta f)$? It is possible to obtain a similar formula for the cross-correlation terms?

Why do we ask these questions? The fact is that new scenarios with new applications are appearing every day within the satellite navigation world. Most of these require very demanding receiver performance (for example, the capability of dealing with degraded scenarios, indoor navigation, and so forth) and new block processing techniques.

In some cases we can approach the study of these new scenarios with a variety of theoretical tools, and the results can eventually be validated by simulation or by real-life experiments. In these cases to have a closed-form expression of the AF, together with its quality of approximation, would greatly help the study — and thus solution — of the problem.

AF formula for $\bar{\tau}=\tau$ and $\bar{f_d}=f_d$

In (1) the signal $y_{IF}(t)$ gives rise to two contributions, which allows us to write the AF in the form

$$S_i(\bar{\tau},\bar{f}_d)=\sum_{m=1}^{N}R_{i,m}(\bar{\tau},\bar{f}_d)=R_{i,i}(\bar{\tau},\bar{f}_d)+\sum_{m\neq i}R_{i,m}(\bar{\tau},\bar{f}_d) \quad (4)$$

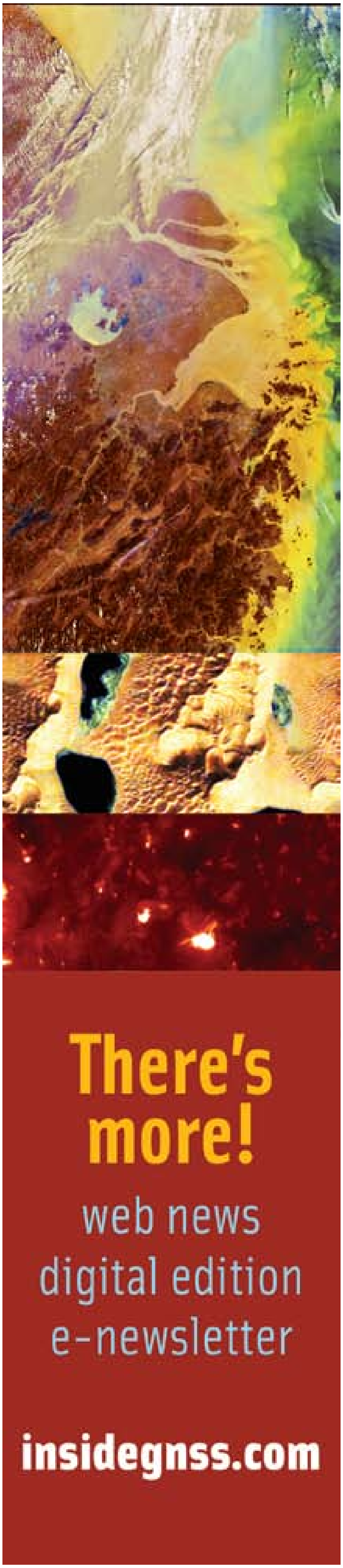
where $R_{i,i}\left(\bar{\tau},\bar{f_d}\right)$ is a term arising from the autocorrelation between the code of the SIS of the i-th satellite and the local code $c_i(t)$, while $R_{i,i}\left(\bar{\tau},\bar{f_d}\right)$, for $m\neq i$, are cross-correlation terms.

Ignoring the data signal (equivalent to integrating within a single navigation data bit), we will now show that the expression in (3) is obtained by approximating the term $R_{i,i}\left(\bar{\tau},\bar{f_d}\right)$ evaluated in the continuous-time domain, that is with the integral

$$R_{i,i}\left(\bar{\tau},\bar{f_d}\right)=\frac{1}{T_d}\int_{-T_d/2}^{T_d/2}\left\{A_i c_i(t-\tau_i)\cos\left[2\pi\left(f_{IF}+f_{d,i}\right)t+\phi_i\right]\right\}\cdot\left\{c_i(t-\bar{\tau})e^{j2\pi\left(f_{IF}+\bar{f_d}\right)t}\right\}dt \quad (5)$$

evaluated over one code period duration (e.g., with an integration time of one millisecond for the GPS C/A code).

The integral is an adequate approximation of the real operation performed by the acquisition systems, that is, the summation in (1). In the case where $\Delta\tau=0$, the local code is perfectly aligned with the incoming code and $c_i(t-\tau)c_i(t-\bar{\tau})|_{\tau=\bar{\tau}}=1$.



By writing the cosine function using the Euler formula (i.e., $\cos\alpha = 0.5(e^{j\alpha} + e^{-j\alpha})$), the product with the local signal contains two sinusoidal terms: one with a frequency of the order of $2f_{IF}$, and the other near baseband:

$$\cos\left[2\pi\left(f_{IF}+f_{d,i}\right)t+\phi_i\right]\cdot e^{j2\pi\left(f_{IF}+\overline{f_d}\right)t}=\frac{1}{2}\left\{e^{j\left[2\pi\left(2f_{IF}+f_{d,i}+\overline{f_d}\right)t+\phi_i\right]}+e^{-j\left[2\pi\left(f_{d,i}-\overline{f_d}\right)t+\phi_i\right]}\right\} \quad (6)$$

The integration of the term at $2f_{IF}$ gives a value which is approximately zero because the average value of a sinusoid is zero. Thus we can write

$$R_{i,i}\left(\overline{\tau},\overline{f_d}\right)\cong\frac{A_i}{2T_d}\int_{-T_d/2}^{T_d/2}e^{-j\left[2\pi\left(f_{IF}+f_{d,i}\right)t+\phi_i\right]}e^{j2\pi\left(f_{IF}+\overline{f_d}\right)t}dt=\frac{A_i}{2}e^{-j\phi_i}\mathrm{Sinc}\left(\Delta f\,T_d\right) \quad (7)$$

Similarly, if $\Delta f \cong 0$, and we neglect again the double frequency contribution, Equation (5) becomes

$$R_{i,m}\left(\overline{\tau},\overline{f_d}\right)=\frac{1}{T_d}\int_{-T_d/2}^{T_d/2}A_i c_i(t-\tau_i)\cos\left[2\pi\left(f_{IF}+f_{d,i}\right)t+\phi_i\right]c_m(t-\overline{\tau})e^{j2\pi\left(f_{IF}+\overline{f_d}\right)t} \quad (8)$$

For $\Delta\tau \cong 0$ and $\Delta f \cong 0$, Equations (6) and (7) can be written in the compact version of Equation (3). This means that the validity region of the approximation in (3) is represented only by the code phase and Doppler shift passing through the AF peak. In this region the approximation is quite good and degrades only in the points where the values given by (3) are of the same order of the terms at the integrate double frequency.

Figure 2 and **Figure 3** compare the AF values obtained by applying (1), for $\overline{\tau} = \tau$ and $\overline{f_d} = f_d$ respectively, with the values given by the mathematical formulas (6) and (7). The relative errors are also represented.

One can easily observe that, within the AF main peak, the approximation error is negligible, while the relative error becomes about 100 percent at the null points.

Moreover, we can observe that for small values of $\Delta\tau$ (that is, $\Delta\tau > 1$ chip duration) Equation (3) also gives a good approximation of the AF, as shown in **Figure 4**.

In contrast, moving farther away from the main peak (i.e., arbitrary values of $\overline{\tau}$ and $\overline{f_d}$) invalidates the approximation. To illustrate, **Figure 5** presents the comparison between the AF and the mathematical formula given by Equation (3) for an arbitrary value of $\Delta\tau$, i.e., $\Delta\tau > 1$ chip duration, namely $\Delta\tau = 50.770\mu s$.

In the next section we will see that a more general formula can be written for the AF, from which (6) and (7) can be derived.

AF formula for arbitrary values of $\overline{\tau}$ and $\overline{f_d}$

We show now that the generic term $R_{i,m}\left(\overline{\tau},\overline{f_d}\right)$ in (4) can also be written in terms of Sinc functions. Starting from the expression

$$R_{i,i}\left(\overline{\tau},\overline{f_d}\right)\cong\frac{A_i}{2T_d}e^{-j\phi_i}\int_{-T_d/2}^{T_d/2}c_i(t-\tau_i)c_i(t-\overline{\tau})dt=\frac{A_i}{2}e^{-j\phi_i}R_{c,i}(\Delta\tau) \quad (9)$$

we observe that the function $b_{i,m}(t) = c_i(t-\tau_i)c_m(t-\overline{\tau})$, for $t(-\infty, \infty)$, is a periodic function with period T_p equal to the code period, and that T_d is generally equal to KT_p, with K integer.

For example, with the GPS C/A code, $K = 20$ is a possible choice if you integrated over an entire 20 millisecond data bit period. At this point, neglecting the $2f_{IF}$ components, and introducing the generic boxcar function

$$p_{T_d}(t)=\begin{cases}1 & |t|<T_d/2\\ 0 & \text{otherwise}\end{cases}$$

the term $R_{i,m}\left(\overline{\tau},\overline{f_d}\right)$ can be written as

$$R_{i,m}\left(\overline{\tau},\overline{f_d}\right)\cong\frac{A_i}{2T_d}e^{-j\phi_i}\int_{-T_d/2}^{T_d/2}b_{i,m}(t)e^{-j2\pi\Delta ft}dt=\frac{A_i}{2T_d}e^{-j\phi_i}\int_{-\infty}^{\infty}p_{T_d}(t)b_{i,m}(t)e^{-j2\pi\Delta ft}dt \quad (10)$$

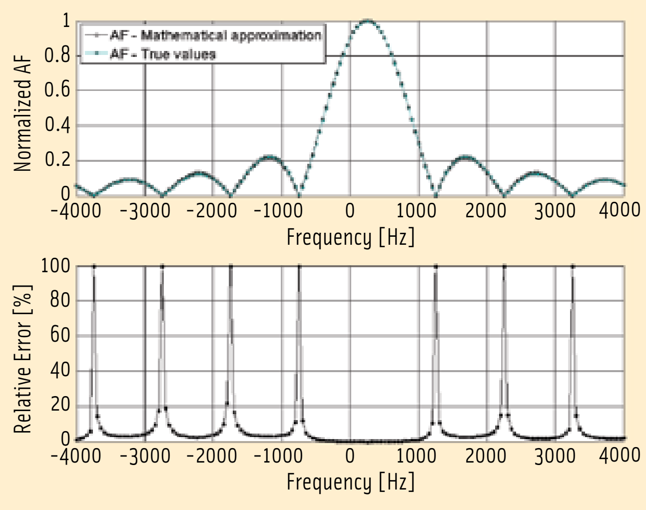


FIGURE 2 Comparison between the AF and its mathematical approximation, given by Equation (6), for $\bar{\tau} = \tau$ (upper curve). Relative error (lower curve)

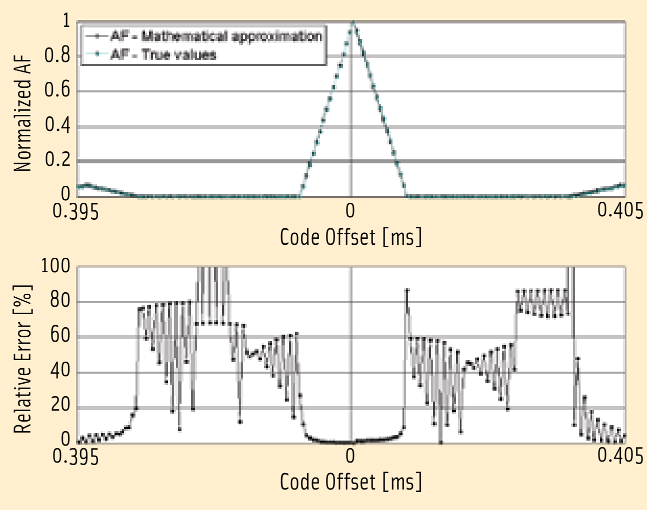


FIGURE 3 Comparison between the AF and its mathematical approximation, given by Eq. (7), for $\bar{f}_d = f_d$ (upper curve); relative error (lower curve)

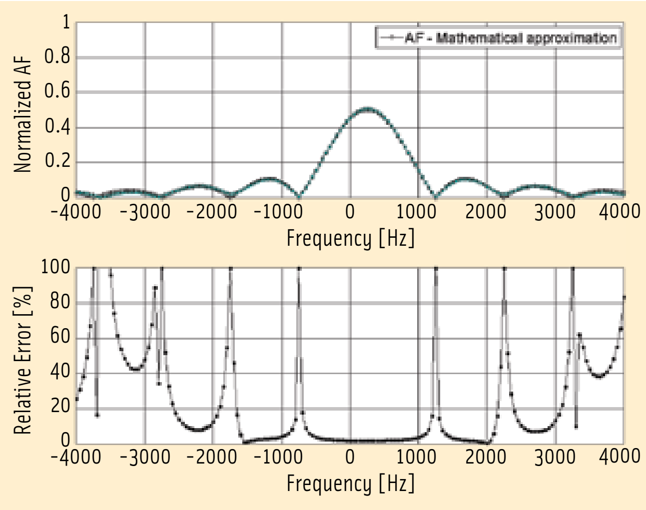


FIGURE 4 Comparison between the AF and its mathematical approximation for small value of $\Delta\tau$, i.e., $\Delta\tau < 1$ chip duration (upper curve). Relative error (lower curve)

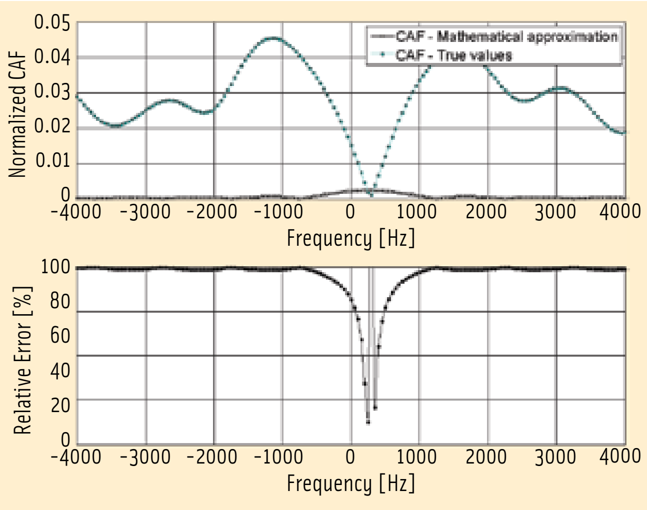


FIGURE 5 Comparison between the AF and its mathematical approximation for a generic value of , i.e., $\Delta\tau$, i.e., $\Delta\tau > 1$ chip duration (upper curve); relative error (lower curve)

The second integral has the structure of a Fourier integral, from which

$$R_{i,m}\left(\bar{\tau}, \bar{f}_d\right) \cong \frac{A_i}{2T_d} e^{-j\phi_i}\left[\mathcal{F}\left\{p_{T_d}(t) b_{i,m}(t)\right\}\right]_{f=\Delta f} = \frac{A_i}{2T_d} e^{-j\phi_i}\left[\mathcal{F}\left\{p_{T_d}(t)\right\} * \mathcal{F}\left\{b_{i,m}(t)\right\}\right]_{f=\Delta f} \qquad (11)$$

where $\mathcal{F}\{\bullet\}$ denotes the Fourier transform, the symbol * convolution, and $\mathcal{F}\left\{p_{T_d}(t)\right\} = T_d \mathrm{Sinc}\left(\Delta f T_d\right)$. The Fourier transform of $b_{i,m}(t)$ is a line spectrum which can be written in terms of the Fourier transform of $b_{i,m}(t)$ taken in the main interval $|t|<T_p/2$.

As we know, the line spectrum of a periodic signal can be given in terms of the Fourier Transform of the waveform in its fundamental period $(-T_p/2, T_p/2)$, which can be written as $b_{i,m}(t)\, p_{T_p}(t)$. Therefore, by introducing the function $B_{i,m}(f) = \mathcal{F}\{b_{i,m}(t)\, p_{T_p}(t)\}$, we obtain

$$\mathcal{F}\left\{b_{i,m}(t)\right\} = \frac{1}{T_p} \sum_{k=-\infty}^{\infty} B_{i,m}\left(\frac{k}{T_p}\right) \delta\left(\Delta f - \frac{k}{T_p}\right) \qquad (12)$$

Mark Petovello is an Assistant Professor in the Department of Geomatics Engineering at the University of Calgary. He has been actively involved in many aspects of positioning and navigation since 1997 including GNSS algorithm development, inertial navigation, sensor integration, and software development.

Email: mark.petovello@ucalgary.ca

By evaluating the following convolution product

$$\mathcal{F}\{p_{T_d}(t)\} * \mathcal{F}\{b_{i,m}(t)\} = \frac{1}{T_p}\sum_{k=-\infty}^{\infty} B_{i,m}\left(\frac{k}{T_p}\right)\text{Sinc}\left[\left(\Delta f - \frac{k}{T_p}\right)T_d\right] \quad (13)$$

it is possible to obtain

$$R_{i,m}(\bar{\tau},\bar{f_d}) \cong \frac{A_i}{2T_p}e^{-j\phi_i}\sum_{k=-\infty}^{\infty} B_{i,m}\left(\frac{k}{T_p}\right)\text{Sinc}\left[\left(\Delta f - \frac{k}{T_p}\right)T_d\right] \quad (14)$$

In the case, when $\Delta\tau = 0$ and $i = m$, $b_{i,m}(t)$, becomes $b_{i,i}(t) = c_i^2(t-\tau_i) = 1$, then only the spectral line for $k = 0$ survives in (13). By integrating over the period T_p we find

$$B(0) = \int_{-T_p/2}^{T_p/2} b_{i,i}(t)dt = T_p.$$

Substituting this result into (13), gives

$$R_{i,i}(\bar{\tau},\bar{f_d}) \cong \frac{A_i}{2}e^{-j\phi_i}\text{Sinc}(\Delta f T_d) \quad (15)$$

which coincides with (6). In the case where $\Delta f = 0$, the Sinc function in Eq. (15) becomes $\text{Sinc}(kT_d/T_p)$.

Since $T_p = KT_p$ and K is an integer, the Sinc function is always equal to zero, except for $k = 0$, in which case

$$R_{i,m}(\bar{\tau},\bar{f_d}) \cong \frac{A_i}{2T_p}e^{-j\phi_i}B_{i,m}(0) = \frac{A_i}{2T_p}e^{-j\phi_i}\int_{-T_p/2}^{T_p/2} c_i(t-\tau_i)c_m(t-\bar{\tau})dt = \frac{A_i}{2}e^{-j\phi_i}R_{c,i,m}(\Delta\tau) \quad (16)$$

where $R_{c,i,m}(\Delta\tau)$ is a normalized cross correlation term. Eq. (15) coincides with (7) when $i = m$.

In conclusion, we can write the AF function given by (8) in the form

$$S_i(\bar{\tau},\bar{f_d}) \cong R_{i,i}(\bar{\tau},\bar{f_d}) + \frac{A_i}{2T_p}e^{-j\phi_i}\sum_{k=-\infty}^{\infty}\text{Sinc}\left[\left(\Delta f - \frac{k}{T_p}\right)T_d\right]\sum_{m\neq i} B_{i,m}\left(\frac{k}{T_p}\right) \quad (17)$$

At this point it is evident that the approximation in (3) is valid in the region of the main peak. In the other points of the search space where the theoretical values are of the same order as the integrated double frequency terms or of the cross terms present in (16), the relative errors become unacceptable.

Finally, from (13) and (16) it appears that the effect of the cross terms is especially concentrated in the points where the Sinc functions exhibit their maxima. With an intelligent and careful design of the codes, working on the cross-spectral lines $\Delta f = k/T_p$, this effect can be mitigated.

BEATRICE MOTELLA AND LETIZIA LO PRESTI

BEATRICE MOTELLA received the Ph. D. in electronics and communications engineering from Politecnico di Torino. Currently, she is a researcher at the Istituto Superiore Mario Boella. Although her studies cover various aspects of GNSS technologies, her main research topic is focused on radio frequency interference monitoring.

LETIZIA LO PRESTI is a full professor with the Information Engineering Faculty of Politecnico di Torino. Her research covers the field of digital signal processing, simulation of telecommunication systems, and the technology of navigation and positioning systems. She is the scientific coordinator of the Master on Navigation and Related Applications held by Politecnico di Torino. IG