GNSS Solutions:

Absolutely Relative Positioning

"GNSS Solutions" is a regular column featuring questions and answers about technical aspects of GNSS. Readers are invited to send their questions to the columnist, **Dr. Mark Petovello**, Department of Geomatics Engineering, University of Calgary, who will find experts to answer them. His e-mail address can be found with his biography at the conclusion of the column.

How do you compute relative positions with GNSS?

GNSS is well recognized as an excellent means of computing position, but many people think that GPS only provides absolute position information. However, GNSS can also provide relative position information. In this column, we will look at some of the details of how this is done.

Definition of Relative

Strictly speaking, all positioning is relative because the process always involves measurements between two or more points. In the context of GNSS, a user's position is always computed relative to the satellite coordinates, which are assumed to be known in an absolute sense. As such, the corresponding position is effectively absolute.

This somewhat pedantic distinction aside, this article focuses on computing the relative positions of GNSS users. To this end, we will consider two situations.

The first involves positioning one object relative to another object. This may include, for example, positioning one receiver relative to a known static receiver. Alternatively, it may include positioning two cars relative to each other to avoid collisions.

The second, and perhaps less well known situation, is positioning one user *over time*. That is, computing the position of a GNSS receiver relative to its position at some previous point in time. This is useful when the motion of the vehicle is the important factor, not its absolute position.

GNSS Measurements

Before looking at how both types of relative positioning can be performed, let us quickly look at the GNSS measurements. For the purpose of this article, the pseudorange measurement made to the *a*-th satellite from the *i*-th receiver can be written as

$$P_i^a = \rho_i^a + b_i + e_i^a \qquad (1)$$

where ρ is the geometric range to the satellite, b is the receiver clock bias (scaled to units of distance), and e is the combination of all measurement errors.

Similarly, the carrier phase measurement, scaled to units of distance by multiplying by the carrier wavelength, can be written as

$$\Phi_i^a = \rho_i^a + b_i + e_i^a + \lambda N_i^a \qquad (2)$$

where λ is the carrier phase wavelength and N is the integer carrier phase ambiguity. We acknowledge, of course, that the measurement errors for the pseudorange and carrier phase cases are not the same. For the time being, however, they can be denoted the same.

Relative Positioning of Two or More Users

To illustrate this concept, we focus only on the pseudorange equations, although the same concepts apply equally to the carrier phase as well. To begin, as in differential GNSS (DGNSS), a between-receiver single difference (denoted Δ) is formed by differencing measurements between receivers to a common satellite.

Applying the result to equation (1) yields

$$\Delta P_{i,j}^a = P_i^a - P_j^a = \Delta\rho_{i,j}^a + \Delta b_{i,j} + \Delta e_{i,j}^a \qquad (3)$$

This is the measurement that will ultimately be used to compute the relative position of receivers *i* and *j*. These measurements are most commonly processed using least-squares or a Kalman filter. Without loss of generality, we only consider the least-squares case in this article.

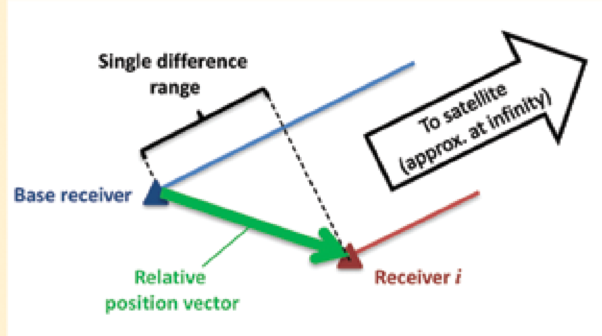

FIGURE 1 Illustration relating single difference measurements to the relative position vector between two points. The single difference measurement is essentially the projection of the relative position vector onto the line of sight vector to the satellite.

Because GNSS measurements are non-linear, corrections to the current estimates of the position and differential clock bias are estimated. We will initially assume that the position of receiver j is already known such that only the coordinates of receiver i need to be estimated (i.e., receiver j is the base station).

In this case, the least-squares corrections to the current estimates is given by

$$\delta\vec{x}_i = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1}\left(\Delta P^a_{i,j} - \Delta\hat{P}^a_{i,j}\right) \quad (4)$$

where H is the Jacobian matrix obtained by computing the partial derivatives of the observations with respect to the unknowns, R is the covariance matrix of the measurement errors, and $\Delta\hat{P}$ is the predicted measurements based on the current estimates of the unknowns.

Consider then, the case where the initial position estimate for receiver i is the known position of the base receiver. In this case, the position correction calculated from equation would be the vector from the base receiver to receiver i. By definition, this is the relative position of the two points.

This follows from Figure 1, where a single difference measurement is shown to be the projection of the relative position vector onto the line of sight vector to the satellite. From this it is not surprising that forming differences results in a relative position solution.

Two assumptions were made in the foregoing development, and we will now address them in the opposite order in which they were introduced. First, we assumed that the initial position estimate was the position of the base receiver. Even if this was not the case, the computed position for receiver i will still be relative to the other point (see Figure 1). In other words, the process of differencing the measurements between receivers automatically leads to relative positioning solutions.

Second, we assumed that the coordinates for the base receiver were known. In fact, this is not an overly restrictive requirement because we do not require this position to be known with a high degree of accuracy. Studies have shown that the amount of error introduced in the *relative* solution is approximately given by

$$\left|\delta\vec{r}_j\right| \cdot \left|\Delta\vec{r}_{i,j}\right| \cdot 10^{-8} \qquad (5)$$

where $\delta\vec{r}_j$ is the error in the assumed-known position of receiver j, and $\Delta\vec{r}_{i,j}$ is the relative position vector between the two points. So, if the position error of the base receiver was 100 meters, for example, the relative position vector could be as large as 10 kilometers before only 1 centimeter of error was introduced into the relative position solution.

This has some very nice practical benefits. First, it means that a single point solution for the base station is more than adequate for computing the relative position of two points for all but very long baselines (i.e., 100s of kilometers). Second, this is what allows GNSS-based attitude determination to work. Specifically, since attitude determination involves determining the relative position of two or more closely spaced receivers (typically within a few meters), the error in the base receiver can be well in excess of one kilometer without any serious effects.

Before moving on, we should note that the same results hold when forming double differences, instead of only between-receiver dingle differences.

Over-Time Relative Positioning

Over-time relative positioning typically uses carrier phase measurements for reasons that will be made clear shortly. As in the previous case, the measurements are also differenced, but in this case the difference is between epochs (denoted d).

Applying this concept to equation (2) gives

$$d\Phi_i^a = \Phi_i^a\left(t_{k+1}\right) - \Phi_i^a\left(t_k\right) = d\rho_i^a + db_i + de_i^a \qquad (6)$$

where the bracketed terms denote the time epoch, and the ambiguity term is removed if no cycle slips have occurred. This latter point is the main reason for using carrier phase measurements, namely, that the ambiguities are differenced out and thus do not need to be resolved to their integer values (a complicated and potentially erroneous process). At the same time, the noise and multipath errors are much smaller than for the pseudorange.

Linearizing equation (6) gives

$$d\Phi_i^a = d\hat{\Phi}_i^a + H_i\left(t_{k+1}\right)\cdot\delta\vec{x}_i\left(t_{k+1}\right) - H_i\left(t_k\right)\cdot\delta\vec{x}_i\left(t_k\right) + d\hat{b}_i + de_i^a \qquad (7)$$

Invitation to Help Out

The GNSS Solutions column is always looking for new contributions. If you would like to prepare a "solution" on a topic that you believe would be well suited to the Q&A format of the column, I would really like to hear about it. This might be a GNSS concept or technique that you believe is often misunderstood, a principle or aspect of GNSS that you often find yourself explaining to others, or perhaps ways in which new and exciting developments can benefit the GNSS community as a whole. I would be happy to work with you to get that information out. As always, questions that you would like to see answered in the column are also welcome.

Mark Petovello, *GNSS Solutions column editor* where H is the Jacobian matrix (same as for the pseudorange case) at the specified time epoch. Because the Jacobian matrix contains unit vectors pointing from the user to the satellites and because these change very slowly due to the large separation between satellite and receiver, we can assume these values to be the same over short time intervals (typically a few seconds, depending on desired accuracy).

Applying this assumption allows us to rewrite equation (7) as

$$\begin{aligned} d\Phi_i^a &= d\hat{\Phi}_i^a + H_i \cdot \left(\delta\vec{x}_i(t_{k+1}) - \delta\vec{x}_i(t_k)\right) + d\hat{b}_i + de_i^a \qquad (8) \\ &= d\hat{\Phi}_i^a + H_i \cdot \delta d\vec{x}_i + d\hat{b}_i + de_i^a \end{aligned}$$

where $\delta\Delta\vec{x}$ is the error (δ) in the over-time relative position of the user ($d\vec{x}$). Equation (8) suggests that the over-time differenced measurements can directly observe the over-time relative position, which is what we set out to do.

Specifically, equation (8) is the basis for a least-squares solution that estimates the relative (over time) position and the over-time clock bias. Details for this are not presented here, but follow directly from the foregoing discussion with the end result being very similar to computing a single-point GNSS solution.

In effect, we are approximating the case in Figure 1, except that instead of two receivers, we are considering the same receiver at two epochs (e.g., the base station is the user at the previous epoch and the receiver is the user at the current epoch).

Summary

In this article we looked at how differencing measurements — either between receivers or over time — led directly to the computation of a relative position. In DGNSS algorithms, if the base station coordinates are known in an absolute coordinate frame, the position of the other receiver is also absolute *only* because it is computed relative to a receiver with an absolute position.

This is often taken for granted, but by recalling the underlying concepts, new applications involving relative GNSS positioning can be envisioned and/or developed. Similarly, differencing measurements over time can provide useful information about the motion of the user, which can be important for many applications. IG

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