GPS EASY Suite II

easy13—RAIM

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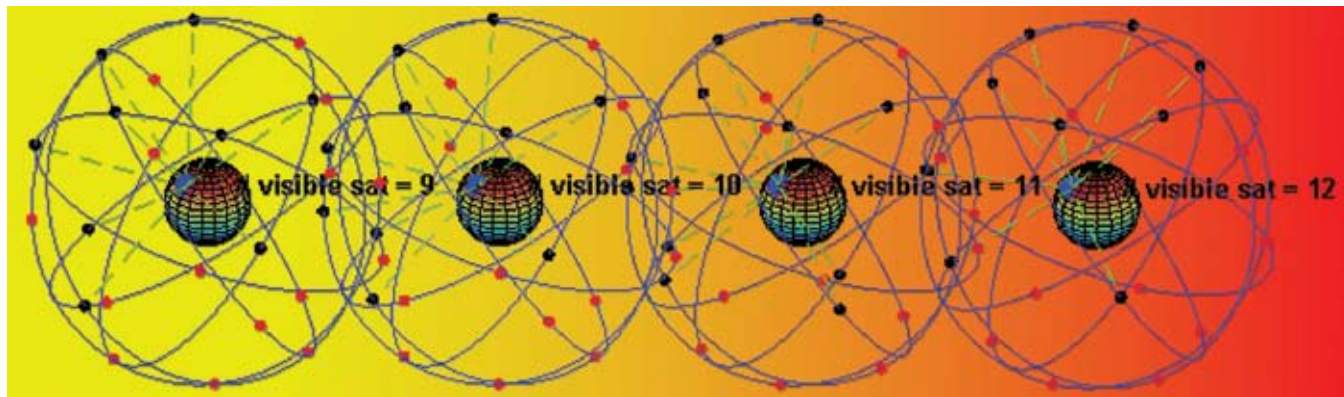
In this installment of the series, the author uses Matlab to illustrate key principles in receiver autonomous integrity monitoring.

The usual formulation of position determination involves four unknowns: three physical dimensions (X,Y,Z) and the satellite-receiver time offset. In cases where we can observe five or more pseudoranges, one might well ask if the redundant pseudoranges could be used to check the consistency among the observations —the fundamental principle behind *receiver autonomous integrity monitoring* (RAIM).

easy13 describes a technique for coping with this situation. In so doing, key concepts such as *horizontal* and *vertical protection levels* (HPL and VPL) are introduced. Necessarily, we also have to introduce some theory that motivates the procedures. (We will return to this topic with some further graphical illustrations in easy14.)

RAIM is a major technique for GNSS in many safety-critical applications. It has been with us since about 1990. Much of the material presented in the following relies on the work by B. Pervan cited in the Additional Resources section near the end of this article.

Let the 4×1 vector of unknowns be denoted $\boldsymbol{x}$, the $m \times 1$ vector of observations be denoted $\boldsymbol{b}$. A is an $m \times 4$ matrix and the pertinent linear observation equation is:

$$A\boldsymbol{x} = \boldsymbol{b} + \boldsymbol{e}, \quad and \quad \Sigma_b = \sigma_b^2 I. \tag{1}$$

The vector $\boldsymbol{e}$ contains residual errors in the observations and Σ_b is the given covariance matrix for the pseudoranges.

RAIM is activated for $m \geq 5$. Presently there is no standardized RAIM method; so, we choose to present the simplest RAIM fault detection based on the residual norm $\|\boldsymbol{e}\|$.

We define the *position error* as

$$\delta \boldsymbol{x} = \boldsymbol{x} - \hat{\boldsymbol{x}} = \boldsymbol{x} - (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}} \boldsymbol{b} = \boldsymbol{x} - (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}} (A\boldsymbol{x} - \boldsymbol{e}) = \tag{2}$$
$$= (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}} \boldsymbol{e}.$$

The estimated residuals $\hat{\boldsymbol{e}}$ equal the observations $\boldsymbol{b}$ minus the estimated observations

$$\hat{\boldsymbol{e}} = \boldsymbol{b} - A\hat{\boldsymbol{x}} = (I - A(A^{\mathrm{T}}A)^{-1} A^{\mathrm{T}})\boldsymbol{b} = S\boldsymbol{b}. \tag{3}$$

The residual vector $\hat{\boldsymbol{e}}$ is in the left nullspace of A. This means $A^{\mathrm{T}}(\boldsymbol{b} - A\hat{\boldsymbol{x}}) = 0$, which are the normal equations. The components of $\hat{\boldsymbol{e}}$ are dependent as they are computed according to (3). The corresponding covariance matrix is

$$\Sigma_{\hat{e}} = S \Sigma_b S^{\mathrm{T}} = \sigma_b^2 S S^{\mathrm{T}} = \sigma_b^2 S . \tag{4}$$

Note that S is a projector and thus idempotent: $S = SS^{\mathrm{T}}$, and that Σ_b is diagonal, while $\Sigma_{\hat{e}}$ is a full matrix!

In order to identify the probability distribution of the residuals we need to transform the vector $\hat{\boldsymbol{e}}$ into independent components. This is done by an old trick, which implies multiplication to the left with $\frac{1}{\sigma_b}W$. The factor W comes from the Cholesky decomposition of the covariance matrix

$$\Sigma_{\hat{e}} = \left(\sigma_b W^{-1}\right)\left(\sigma_b W^{-\mathrm{T}}\right).$$

The transformed residual vector

$$\hat{\boldsymbol{e}}^* = \frac{1}{\sigma_b} W\hat{\boldsymbol{e}}$$

has independent components.

Under normal conditions (small $\|\hat{e}\|$) the weighted sum of squares is

$$\hat{e}^{\mathrm{T}} \frac{1}{\sigma_b^2} W^{\mathrm{T}} W \hat{e} = (\hat{e}^*)^{\mathrm{T}} \hat{e}^*. \tag{5}$$

The vector $\hat{e}^*$ is gaussian and independent and identically distributed with zero mean and variance 1:

$$\|\hat{e}^*\| : \chi_{m-4}^2, \quad m > 4. \tag{6}$$

Figure 1 graphically illustrates some basic RAIM states, which eventually (as illustrated in Figure 4) become four possible outcomes or "cases" experienced when using RAIM techniques: normal (detected) error condition (NC), missed detection (MD), detection failure (DF), and false alarm (FA).

A residual threshold can be set analytically using (6) to achieve any desired probability of false alarm under normal error conditions:

$$P(\mathrm{FA} \,|\, \mathrm{NC}) = P(\|\hat{e}^*\| > R \,|\, \mathrm{NC}) = \tag{7}$$

$$= \frac{1}{2^{(m-4)/2}\, \Gamma\left(\frac{m-4}{2}\right)} \int_{R^2/\sigma_b^2}^{\infty} s^{\left(\frac{m-4}{2}-1\right)} e^{-s/2}\, ds.$$

Given the values of $m - 4$ and $P(\mathrm{FA} \,|\, \mathrm{NC})$ we may solve (7) for the residual threshold R. The situation is depicted in **Figure 2**. The Matlab code (*M*-code) for plotting this figure is contained in the file "fap.m," which can be found at the easy2 webpage < http://kom.aau.dk/~borre/easy2/easy13>.

In Figure 1 a horizontal line constraint is drawn to represent the protection level *a*. Note that, for small failure magnitudes, it is possible for the accuracy specification not to be breached.

In the event that the position error δx exceeds a predefined protection level *a*, but $\|\hat{e}\| < R$ — the inside threshold R determined from equation (7) — a *missed detection* has occurred, case II, see Figures 1 and 4. The corresponding probability is defined as

$$P(\mathrm{MD}) = P(\|\hat{e}\| < R, \|\delta x\| > a). \tag{8}$$

In general a condition between $\|\hat{e}\|$ and $\|\delta x\|$ will exist. We must quantify the degree of this correlation in order to demonstrate the integrity monitoring capability of RAIM-based fault detection. The result is given later in equation (13).

According to equation (2) the residual $\hat{e}$ and the position error δx will scale proportionally, with the factor $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. Hence, the normal condition (NC) confidence ellipse will slide up the *failure mode axis* with slope α.

In (2) the position error

$$\delta x = (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}} e$$

is defined in a 3-D Cartesian coordinate system (X,Y,Z). However, for practical use a local topocentric coordinate system $\delta x_{ENU} = (e, n, u)$ is more appropriate. A position vector at (φ, λ, h) given in the (X,Y,Z) system can be transformed into the east, north, up (e, n, u) system through multiplication by the orthogonal transformation matrix F:
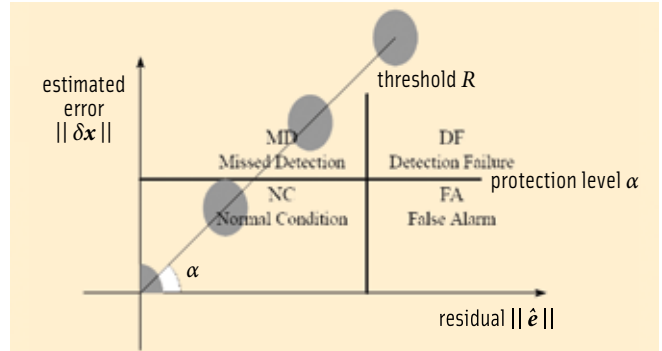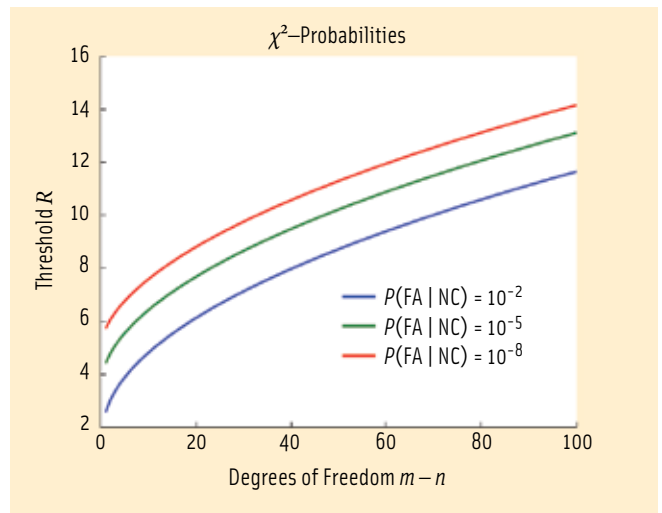

FIGURE 1 Basic RAIM states

FIGURE 2 Probability of a false alarm

$$F = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{bmatrix}. \tag{9}$$

In the following discussion we only consider the three coordinates (X,Y,Z). So we delete the last column of A and get a new matrix $A_0 = A(:, 1:3)$. Similarly we delete the last element of δx and define $\delta x_0 = \delta x(1:3)$; hence:

$$\delta x_{ENU} = \begin{bmatrix} e \\ n \\ u \end{bmatrix} = F \delta x_0 = F (A_0^{\mathrm{T}} A_0)^{-1} A_0^{\mathrm{T}} e = M e. \tag{10}$$

Note that rows 1 and 2 of the $3 \times m$ matrix M relate to easting and northing. Finally we define $M_0 = M(1:2, 1:2)$.

Imagine now a *failure* of magnitude β in satellite i (β is placed as the ith component):

$$e = \begin{bmatrix} 0 & \cdots & 0 & \beta & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}. \tag{11}$$

We compute the norm squared for this special choice of e:

$$\|\delta x_{EN}\|^2 = e^{\mathrm{T}} M_0^{\mathrm{T}} M_0 e.$$

The many zeros in e simplifies this to

$$\left\|\delta\boldsymbol{x}_{EN}\right\|^2 = (m_{1i}^2 + m_{2i}^2)\beta^2.$$

From (3) we recall $\hat{\boldsymbol{e}} = S\boldsymbol{b}$ or

$$\left\|\hat{\boldsymbol{e}}\right\|^2 = \hat{\boldsymbol{e}}^T\hat{\boldsymbol{e}} = \boldsymbol{b}^T S^T S\boldsymbol{b} = s_{ii}\beta^2$$

as $S^T S = S$. The diagonal entry (i,i) of S is called s_{ii}. Now

$$\left\|\delta\boldsymbol{x}_{EN}\right\|^2 = \frac{m_{1i}^2 + m_{2i}^2}{s_{ii}}\left\|\hat{\boldsymbol{e}}\right\|^2$$

or

$$\left\|\delta\boldsymbol{x}_{EN}\right\| = \sqrt{\frac{m_{1i}^2 + m_{2i}^2}{s_{ii}}}\left\|\hat{\boldsymbol{e}}\right\| = \alpha_i\left\|\hat{\boldsymbol{e}}\right\|. \tag{12}$$

This is the equation for a straight line through the origin and with slope α_i. The slope α_i of the failure mode axis related to satellite i is computed as

$$\alpha_i = \sqrt{\frac{m_{1i}^2 + m_{2i}^2}{s_{ii}}}. \tag{13}$$

The slope values are computed for all $i = 1, \ldots, m$, and the corresponding lines are depicted in **Figure 3**. The PRNs in this figure are the ones included in easy2 (computation of a satellite's position from an ephemeris).

The likelihood that the RAIM algorithm may detect an observational error depends on the satellite geometry. A poor geometry does not necessarily indicate observational errors, but if errors are present they may be difficult to detect.

The slope α_i provides a measure of the difficulty in accurately detecting a fault in presence of noise: the larger the slope, the more difficult it is to detect the fault.

The failure mode axes in Figure 3 through the origin with slope α_i are given exclusively from the geometry determined by the satellites and the receiver. The mode axis with maximum value of α_i is called α_{max} and the HPL is defined as

$$HPL = \alpha_{max}\sigma_0 \tag{14}$$

where σ_0 is the standard deviation of the pseudoranges

$$\sigma_0 = \sqrt{\frac{\hat{\boldsymbol{e}}^T\Sigma_b^{-1}\hat{\boldsymbol{e}}}{m-4}}.$$

At <http://www.nstb.tc.faa.gov/Terms.html> readers may find the following definition of HPL: The Horizontal Protection Level is the radius of a circle in the horizontal plane with its center being at the true position which describes the region that is assured to contain the indicated horizontal position.

The resulting RAIM fault detection algorithm is a simple one: Check the residual statistic to see if it is larger than the threshold R. If so, a system failure is declared. Given this simple algorithm, four outcomes are possible, refer to Figures 1 and 4.

Under a *normal condition* (NC), the position error $\|\delta\boldsymbol{x}\|$ does not exceed the protection level a, and the residual is smaller than the threshold R, as in case III. If the position error does not exceed the protection level a, but the residual is larger than
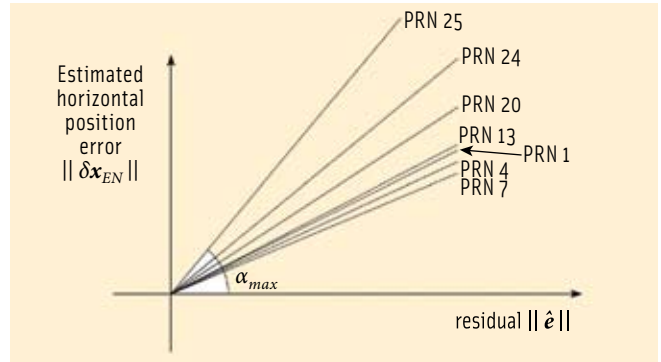
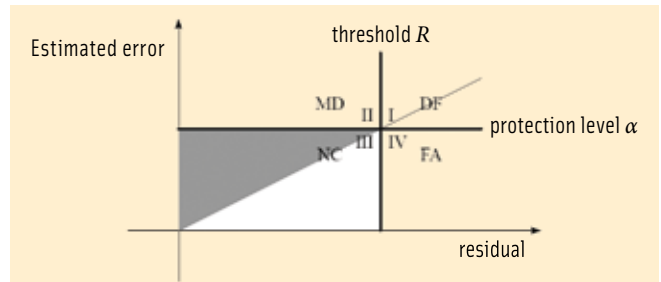FIGURE 3 **Characteristic slopes for seven visible satellites**

FIGURE 4 **RAIM status, repeated. We aim at having plots in the gray area.**

the threshold R, a *false alarm* (FA) has occurred, which is case IV. When both protection level and residual threshold have been breached, a *detection failure* (DF) has occurred — case I. Finally, a *missed detection* (MD) happens when the position error $\|\delta\boldsymbol{x}\|$ is larger than the protection level a, but the residual is smaller than the threshold R; that is case II.

In the general case, of course, more than one failure mode exists, that is, e in (11) has more than one non-zero component. However, this presentation does not deal with that case. In their book (see Additional Resources), R. G. Brown and P. Y. C. Hwang investigate RAIM in case of non-uniform weighted observations and multiple faults.

Because the horizontal protection level depends on satellite geometry, it must be computed for each epoch and each position. If the HPL is below the protection level, RAIM is said to be available for that epoch.

Referring to **Figure 4**, we may introduce inequalities, which characterize each region in the figure. For notational reasons, in addition to the predefined protection level a, we introduce the obvious new variables $x = \|\hat{\boldsymbol{e}}\|$ (norm of residuals) and $y = \|\delta\boldsymbol{x}_{EN}\|$ (horizontal position error):

case I	upper part	$a < y < x$
	lower part	$a < x < y$
case II		$y < a < x$
case III	upper part	$y < x < a$
	lower part	$x < y < a$
case IV		$x < a < y$

A desired result will plot in the case III area, upper part, which is called *normal operation*. The foregoing description may be summarized into the following procedure for determining HPL:

Input to RAIM: The variance σ_b^2 of a pseudorange observation, the coefficient matrix A of the linearized least-squares observation equations, and the maximum allowable probabilities for a false alarm $P(\text{FA})$ and a missed detection $P(\text{MD})$.

Output of the algorithm: Horizontal protection level (HPL), which is the radius of a circle, centered at the true position that is assured to contain the indicated horizontal position with the given probability of false alarm and missed detection.

Similarly for Vertical Protection Level (VPL). Again, an official definition (at <http://www.nstb.tc.faa.gov/Terms.html>) is: The vertical protection level is half the length of a segment on the vertical axis with its center being at the true position, which describes the region that is assured to contain the indicated vertical position.

Additional Resources

The original Easy Suite can be found online at <http://kom.aau.dk/~borre/easy/>. The complete set of Easy Suite II Matlab codes can be found in compressed ("zipped") files at <http://gps.aau.dk/~borre/easy2>.

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Author

Originally graduated as a chartered surveyor, **Kai Borre** obtained his Ph.D. in geodesy from Copenhagen University, and a Doctor of Technology degree from Graz University of Technology. He has been a full professor in geodesy at Aalborg University since 1976.

For more than 30 years Borre has conducted research and performed education in the area of satellite based positioning. In 1996 he established the Danish GPS Center and since 2000 has been head of a two-year international M.Sc. program in GPS technology.

Borre is a coauthor of the widely used textbooks, *Linear Algebra, Geodesy, and GPS* and *A Software-Defined GPS* and *Galileo Receiver; Single-Frequency Approach*.

Since the early 1990s Borre has published Matlab code for processing of GPS observations, and since 2003 he published Matlab code together with explanatory text, a very successful new pedagogical concept. **IG**